1 Playing with waves in $1+1$ dimensions

Let us derive some simple results on the wave equation in $1+1$ dimensions

$$\left(\frac{\partial^2}{\partial t^2} - u^2 \frac{\partial^2}{\partial x^2}\right) \psi(t,x) = 0 \tag{1}$$

($u > 0$ is the propagation speed). Changing coordinates to

$$\begin{aligned} \xi &= x + ut \\ \eta &= x - ut \end{aligned} \tag{2}$$

it is easy to show that, in terms of these new variables, the wave equation reads

$$\frac{\partial^2}{\partial \xi \partial \eta} \psi(\xi, \eta) = 0 \tag{3}$$

and therefore that

$$\psi(\xi, \eta) = f_-(\xi) + f_+(\eta) \tag{4}$$

i.e. ψ is a sum of signals propagating in the negative and positive directions of the x axis. To see this, we notice that

$$\xi_1 = \xi_2 \Rightarrow f_-(\xi_1) = f_-(\xi_2) \tag{5}$$

and therefore the function has the same value whenever

$$x_1 + ut_1 = x_2 + ut_2 \tag{6}$$

that is, when

$$\frac{x_2 - x_1}{t_2 - t_1} = -u < 0 \tag{7}$$

describing a point particle moving with a constant negative velocity along the x axis. Same argument applies to f_+.

2 Separation of variables?

Let us look for a solution of the form

$$\psi(t,x) = \phi(t)\rho(x) \tag{8}$$

which gives

$$\rho \frac{d^2}{dt^2}\phi = u^2 \phi \frac{d^2}{dx^2}\rho \tag{9}$$

and therefore

$$\frac{1}{\phi}\frac{d^2}{dt^2}\phi = \frac{u^2}{\rho}\frac{d^2}{dx^2}\rho \tag{10}$$

Now, the left hand side is a function of t only, while the right hand side is a function of x only: they must be both equal to a constant which has the dimensions of an frequency squared:

$$\frac{1}{\phi}\frac{d^2}{dt^2}\phi = \pm\omega^2 \tag{11}$$

$$\frac{u^2}{\rho}\frac{d^2}{dx^2}\rho = \pm\omega^2 \tag{12}$$

Which we can solve with the usual techniques (exponentials or oscillators according to the sign we choose for the right hand side). We already see that our solutions are not always good. If for instance we take the initial condition

$$\psi(t=0,x) = \exp\left(-\frac{x^2}{2L^2}\right) \tag{13}$$

we get the solution of the wave equation (change reference frame!)

$$\psi(t,x) = \exp\left(-\frac{(x-ut)^2}{2L^2}\right) \tag{14}$$

that is obviously incompatilbe with the separation of variable hypothesis. It is also interesting to check that

$$\psi(\xi,\eta) = \phi(\xi+\eta)\rho(\xi-\eta) \tag{15}$$

is consistent with the wave equation

$$\frac{\partial^2}{\partial\xi\partial\eta}\psi(\xi,\eta) = 0$$